Connectivity, Rigidity and Online Decentralized Maintenance Methods

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Partial list:

- P. Yang, R.A. Freeman, G.J. Gordon, K.M. Lynch, S.S. Srinivasa, and R. Sukthankar, "Decentralized estimation and control of graph connectivity for mobile sensor networks," Automatica, vol. 46, no. 2. pp. 390–396, Feb. 2010.
- G. Hollinger and S. Singh, "Multirobot coordination with periodic connectivity: Theory and experiments," IEEE Transactions on Robotics , 2012,
- L. Sabattini, C. Secchi, N. Chopra, and A Gasparri. Distributed Control of Multirobot Systems With Global Connectivity Maintenance. Robotics, IEEE Transactions on Robotics, 29(5):1326-1332, 2013.
- D. Carboni, R.K. Williams, A. Gasparri, G. Ulivi, and G.S. Sukhatme. Rigidity-Preserving Team Partitions in Multi-Agent Networks. IEEE Transactions on Cybernetics, pp 1-14, 2014.

If you want to know more about what follows:

- Robuffo Giordano, P., A. Franchi, C. Secchi, and H. H. Bülthoff (2013). *A Passivity-Based Decentralized Strategy for Generalized Connectivity* Maintenance".
The International Journal of Robotics Research 32.3, pp. 299–323.

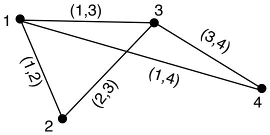
- Zelazo, D., A. Franchi, H. H. Bülthoff, and P. Robuffo Giordano (2014). *Decentralized Rigidity Maintenance Control with Range Measurements for Multi-Robot Systems*.
The International Journal of Robotics Research 34.1, pp. 105–128.

- Nestmeyer T., P. Robuffo Giordano, H. H. Bülthoff, and A. Franchi, *Decentralized Simultaneous Multi-target Exploration using a Connected Network of Multiple Robots*.
Under Review.

Graphs, Matrices, and Eigenvalues

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is an **undirected graph** or simply **graph**

- $\mathcal{V} = \{1, \dots, N\}$ **vertex set**
- $\mathcal{E} \subset (\mathcal{V} \times \mathcal{V})/\sim$ **edge set**
- $\sim$ equivalence relation identifying (i, j) and (j, i)



A Graph models an Adjacency Structure

$[(i, j)] \in \mathcal{E} \Leftrightarrow$ vertexes i and j are **neighbors** or **adjacent**

- (i, j), $i < j$ representative element of the **equivalence class** $[(i, j)]$

$$[\mathcal{V} \times \mathcal{V}] = \{(1, 2),\ (1, 3), \dots, (1, N), \dots, (N-1, N)\}$$
$$= \{e_1,\ e_2, \dots e_{N-1}, \dots, e_{N(N-1)/2}\}$$

- $[(i, i)] \notin \mathcal{E}$, $\forall i \in \mathcal{V}$ (**no self-loops**)
- $\mathcal{N}_i = \{j \in \mathcal{V} \mid (i, j) \in \mathcal{E}\}$ set of **neighbors** of i

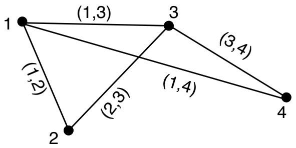
$E \in \mathbb{R}^{N \times N(N-1)/2}$ is the (full) **incidence matrix** of $\mathcal{G}$

$\forall e_k = (i, j) \in [\mathcal{V} \times \mathcal{V}]$:

- $E_{ik} = -1$ and $E_{jk} = 1$, if $e_k \in \mathcal{E}$
- $E_{ik} = 0$ and $E_{jk} = 0$, otherwise

Matricial representation of a graph

Example:



$$E = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & -1 \end{pmatrix}$$

$$e_1 \quad e_2 \quad e_3 \quad e_4 \quad e_5 \quad e_6$$

remember:
$\{e_1,\ e_2, \ldots, e_{N-1}, \ldots, e_{N(N-1)/2}\} = \{(1,\ 2),\ (1,\ 3), \ldots, (1,\ N), \ldots (N-1,\ N)\}$

Assume N mobile robots moving in an environment:

- $\mathbf{x}_i \in \mathbb{R}^{N_\mathbf{x}}$ **i-th robot configuration**, $i \in 1 \dots N$
- $\mathbf{z} \in \mathbb{R}^{N_\mathbf{z}}$ **environment configuration**

Consider two maps

$$\textbf{robot map} \quad \mathbf{v}: \quad \mathbb{R}^{N_\mathbf{x}} \ni \mathbf{x}_i \mapsto \mathbf{v}(\mathbf{x}_i) = \mathbf{v}_i \in \mathbb{R}^{N_\mathbf{v}}$$

$$\textbf{connection map} \quad \mathbf{w}: \quad \mathbb{R}^{N_\mathbf{x}} \times \mathbb{R}^{N_\mathbf{x}} \times \mathbb{R}^{N_\mathbf{z}} \ni (\mathbf{x}_i, \mathbf{x}_j, \mathbf{z}) \mapsto \mathbf{w}(\mathbf{x}_i, \mathbf{x}_j, \mathbf{z}) = \mathbf{w}_{ij} \in \mathbb{R}_{\geq 0}$$

with the properties

- $\mathbf{w}_{ij} = \mathbf{w}_{ji}$ (symmetry)
- $\mathbf{w}_{ii} = 0$

example: what can those maps model?

Associated Graph and Framework

The connection map **w** defines an **associated graph** $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where

- $\mathcal{V} = \{1, 2, \ldots, N\}$
- $\mathcal{E} = \{e_k = (i, j) \mid \mathbf{w}_{ij} > 0\}$
- the **positive weight** $\mathbf{w}_{ij}$ is associated to each edge $(i, j) \in \mathcal{E}$

Both maps **v** and **w** define an **associated framework** $(\mathcal{G}, \mathbf{v})$ where

- $\mathcal{G}$ is the associated graph
- $\mathbf{v}_i$ is associated to each vertex $i \in \mathcal{V}$

$$A = \begin{pmatrix} 0 & \mathbf{w}_{12} & \dots & \mathbf{w}_{1N} \\ \mathbf{w}_{12} & 0 & \dots & \mathbf{w}_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{w}_{1N} & \mathbf{w}_{2N} & \cdots & 0 \end{pmatrix} \in \mathbb{R}^{N \times N}$$ is the **adjacency** (or **weight**) matrix of $\mathcal{G}$

Note that

- $A_{ij} = 0$ if $(i, j) \notin \mathcal{E}$
- $A_{ij} > 0$ otherwise

Properties:

P.1 $A = A(\mathbf{x}_1, \dots, \mathbf{x}_N, \mathbf{z})$

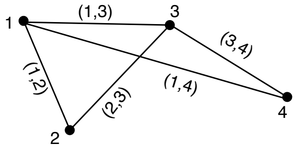
P.2 A is square

P.3 $A_{ij} = A_{ij}$ (symmetric)

P.4 $A_{ij} = A_{ij} \geq 0$ (nonnegative)

P.5 $A_{ii} = 0$

Example:



$$A = \begin{pmatrix} 0 & \mathbf{w}_{12} & \mathbf{w}_{13} & \mathbf{w}_{14} \\ \mathbf{w}_{12} & 0 & \mathbf{w}_{23} & 0 \\ \mathbf{w}_{13} & \mathbf{w}_{23} & 0 & \mathbf{w}_{34} \\ \mathbf{w}_{14} & 0 & \mathbf{w}_{34} & 0 \end{pmatrix}$$

$$L = \begin{pmatrix} \sum_{j=1}^{n} \mathbf{w}_{1j} & -\mathbf{w}_{12} & \dots & -\mathbf{w}_{1N} \\ -\mathbf{w}_{12} & \sum_{j=1}^{n} \mathbf{w}_{j2} & \dots & -\mathbf{w}_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -\mathbf{w}_{1N} & -\mathbf{w}_{2N} & \dots & \sum_{j=1}^{n} \mathbf{w}_{jN} \end{pmatrix} \in \mathbb{R}^{N \times N} \text{ is the } \textbf{Laplacian} \text{ matrix of } \mathcal{G}$$

Note that

- $L = \mathrm{diag}(\delta_i) - A$,

  where $\delta_i = \sum_{j=1}^{n} \mathbf{w}_{ij}$ (degree of vertex i)

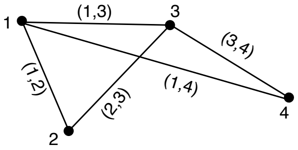
Properties:

P.1 $L = L(\mathbf{x}_1, \dots, \mathbf{x}_N, \mathbf{z})$

P.2 L is square

P.3 $L_{ij} = L_{ij}$ (symmetric)

Example:



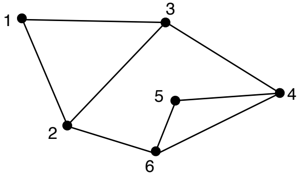
$$L = \begin{pmatrix} \mathbf{w}_{12}+\mathbf{w}_{13}+\mathbf{w}_{14} & -\mathbf{w}_{12} & -\mathbf{w}_{13} & -\mathbf{w}_{14} \\ -\mathbf{w}_{12} & \mathbf{w}_{12}+\mathbf{w}_{23} & -\mathbf{w}_{23} & 0 \\ -\mathbf{w}_{13} & -\mathbf{w}_{23} & \mathbf{w}_{13}+\mathbf{w}_{23}+\mathbf{w}_{34} & -\mathbf{w}_{34} \\ -\mathbf{w}_{14} & 0 & -\mathbf{w}_{34} & \mathbf{w}_{14}+\mathbf{w}_{34} \end{pmatrix}$$

Connectivity

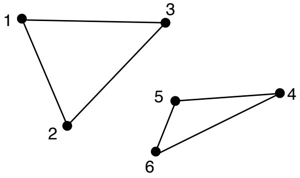
$\mathcal{G}$ is **connected** if there is a **path** between every pair of vertices, i.e.,

$$\forall i \in \mathcal{V} \text{ and } j \in \mathcal{V} \backslash i, \quad \exists \text{ a path (sequence of adjacent edges) from } i \text{ to } j$$

This is a **combinatorial definition** of connectivity



connected graph



disconnected graph

question: connectivity is a global property, what does it mean? and why it is global?

Importance of Connectivity

What connectivity can model?

- connected **communication** network
- connected **sensing** network
- connected **control** network
- connected **planning** roadmap

What connectivity is important for?

- pass a message from **any** robot **to any** other robot
- know the relative position between any two robots in a **common frame**
- converge to a **common point**
- **share** a common goal

Related concepts

- group, cohesiveness
- aggregation
- sharing

Additional properties of $L = \mathrm{diag}(\delta_i) - A$

- L is **positive semi-definite**, i.e., all the eigenvalues are real and non-negative

$$0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$$

- $\sum_{j=1}^{n} L_{ij} = 0 \quad \forall i = 1 \ldots N$, i.e., $L\mathbf{1} = \mathbf{0}$, therefore

$$\lambda_1 = 0 \text{ and it is associated to the eigenvector } \mathbf{1} = \begin{pmatrix} 1 & 1 & \ldots & 1 \end{pmatrix}^T$$

(Fiedler 1973)

$\lambda_2 > 0$ if the graph $\mathcal{G}$ is **connected** and $\lambda_2 = 0$ otherwise

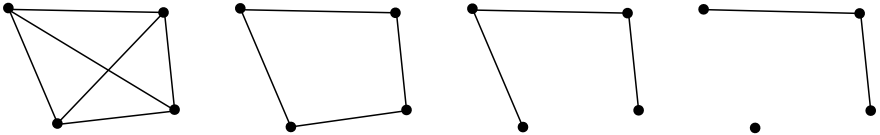
Spectrum of the Laplacian Matrix and Algebraic Connectivity

λ_2 provides an **algebraic definition** of connectivity

$\Rightarrow$ λ_2 is called *algebraic connectivity*, *connectivity eigenvalue*, or **Fiedler eigenvalue**

$\lambda_2 = \lambda_2(\mathbf{x}_1, \ldots, \mathbf{x}_N, \mathbf{z})$ is a **global** quantity

Example (if $\mathbf{w}_{ij} \in \{0, 1\}$):



$\lambda_2 = 4$ $\lambda_2 = 2$ $\lambda_2 = 0.58$ $\lambda_2 = 0$

Framework of Positions

A **framework of positions** is a particular framework $(\mathcal{G}, \mathbf{v})$ in the special case in which $\mathbf{v} : \mathcal{V} \to \mathbb{R}^d$ maps each vertex to the position in $\mathbb{R}^d$ of the i-th robot

- if $d = 2$, $\mathbf{v}_i = \mathbf{p}_i = \begin{pmatrix} p_i^x \\ p_i^y \end{pmatrix}$, **2D position** of robot i
- if $d = 3$, $\mathbf{v}_i = \mathbf{p}_i = \begin{pmatrix} p_i^x \\ p_i^y \\ p_i^z \end{pmatrix}$, **3D position** of robot i

In the following

- it will be (mainly) $d = 3$, similar results apply for $d = 2$
- we refer only to framework of positions, called simply frameworks

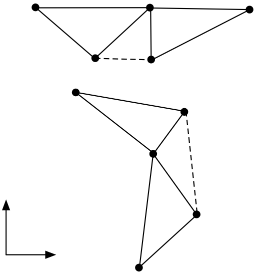
Equivalent and Congruent Frameworks

Consider two frameworks $(\mathcal{G}, \mathbf{p}')$ and $(\mathcal{G}, \mathbf{p}'')$

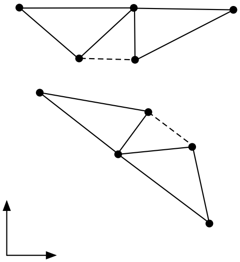
- same graph $\mathcal{G}$
- different positions $\mathbf{p}'$ and $\mathbf{p}''$

Frameworks $(\mathcal{G}, \mathbf{p}')$ and $(\mathcal{G}, \mathbf{p}'')$ are

- **equivalent**: if $\|\mathbf{p}_i' - \mathbf{p}_j'\| = \|\mathbf{p}_i'' - \mathbf{p}_j''\|$ for all $(i, j) \in \mathcal{E}$, and
- **congruent**: if $\|\mathbf{p}_i' - \mathbf{p}_j'\| = \|\mathbf{p}_i'' - \mathbf{p}_j''\|$ for all $(i, j) \in \mathcal{V} \times \mathcal{V}$



equivalent frameworks



congruent frameworks

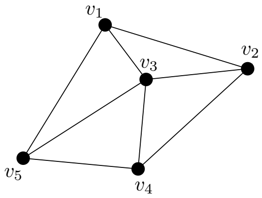
Global Rigidity

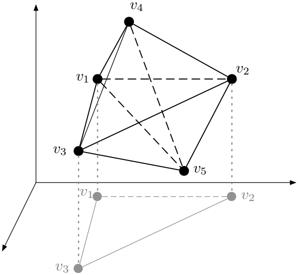
The framework $(\mathcal{G}, \mathbf{p}')$ is **globally rigid** if every other framework $(\mathcal{G}, \mathbf{p}'')$ which

- is equivalent to $(\mathcal{G}, \mathbf{p}'')$

is also congruent to $(\mathcal{G}, \mathbf{p}')$

This is, again, a **combinatorial definition**





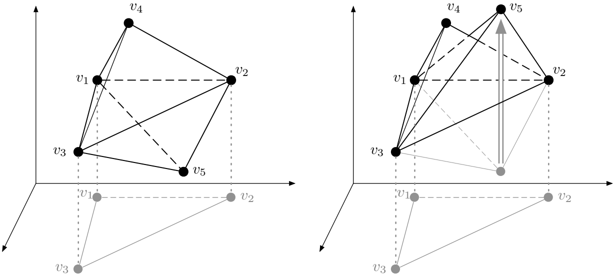
Rigidity

The framework $(\mathcal{G}, \mathbf{p}')$ is **rigid** if $\exists \epsilon > 0$ such that every other framework $(\mathcal{G}, \mathbf{p}'')$ which

- is equivalent to $(\mathcal{G}, \mathbf{p}'')$ and
- satisfies $\|\mathbf{p}_i' - \mathbf{p}_i''\| < \epsilon$ for all $i \in \mathcal{V}$,

is congruent to $(\mathcal{G}, \mathbf{p}')$

This is, again, a **combinatorial definition**



question: is rigidity a global property of the graph as well?

Importance of Rigidity

What rigidity can model?

- rigid **mechanical structure** made of **bars**

  but also:

- rigid **sensing network**
- rigid **control network**

What rigidity is important for?

- **univocally** compute the arrangement (**shape**) of a group of robots only using **inter-distances**
- achieve (or track) a desired shape **only controlling** the **inter-distances** (formation control)

Related concepts

- parallel rigidity
- persistent graph
- tensegrity

question: do you know an example of use of rigidity in robotics?

question: do you know an example of use of rigidity in robotics?

6-DOF **Stewart platform** parallel robot

Credits: Robert L. Williams II

Let's give a definition of rigidity that is differential ($\Leftrightarrow$ involves **infinitesimal motions**)

Consider a trajectory $\mathbf{p}(t)$ with $t \geq t_0$ and impose **equivalence** along the trajectory:

$$\|\mathbf{p}_i(t) - \mathbf{p}_j(t)\|^2 = \|\mathbf{p}_i(t_0) - \mathbf{p}_j(t_0)\|^2 = \text{const} \quad \text{for all} \quad (i,j) \in \mathcal{E}, \;\; \forall t \geq t_0$$

Differentiating with respect to time the constraint above:

$$\left(\mathbf{p}_i(t) - \mathbf{p}_j(t)\right)^T \left(\dot{\mathbf{p}}_i(t) - \dot{\mathbf{p}}_j(t)\right) = 0 \quad \text{for all} \quad (i,j) \in \mathcal{E}, \;\; \forall t \geq t_0 \tag{1}$$

Trivial Motion

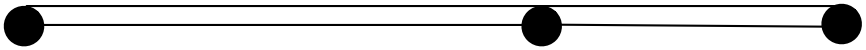
A collective motion that consists of only **global roto-translations** of the whole set of positions in the framework

Infinitesimal Rigidity

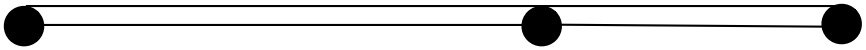
The framework $(\mathcal{G}, \mathbf{p}(t_0))$ is **infinitesimally rigid** if every possible motion that satisfies (1) is **trivial**

Rigidity vs Infinitesimal Rigidity

question: is this framework rigid in $\mathbb{R}^2$? is it infinitesimally rigid?



question: is this framework rigid in $\mathbb{R}^2$? is it infinitesimally rigid?



- infinitesimal rigidity $\Rightarrow$ rigidity
- rigidity $\not\Rightarrow$ infinitesimal rigidity

Matricial Representation of Infinitesimal Rigidity

Let us write the infinitesimal rigidity constraint in a matricial form

$$\left(\mathbf{p}_i(t) - \mathbf{p}_j(t)\right)^T \left(\dot{\mathbf{p}}_i(t) - \dot{\mathbf{p}}_j(t)\right) = 0 \quad \text{for all} \quad (i,j) \in \mathcal{E}, \;\; \forall t \geq t_0$$

$$\Updownarrow$$

$$\mathbf{w_{ij}} \left(\mathbf{p}_i(t) - \mathbf{p}_j(t)\right)^T \left(\dot{\mathbf{p}}_i(t) - \dot{\mathbf{p}}_j(t)\right) = 0 \quad \text{for all} \quad e_k = (i,j) \in [\mathcal{V} \times \mathcal{V}], \;\; \forall t \geq t_0$$

$$
\begin{aligned}
0 &= \mathbf{w}_{ij} \left(\mathbf{p}_i(t) - \mathbf{p}_j(t)\right)^T \left(\dot{\mathbf{p}}_i(t) - \dot{\mathbf{p}}_j(t)\right) = \\
&= \mathbf{w}_{ij} \left(\mathbf{p}_i(t) - \mathbf{p}_j(t)\right)^T \dot{\mathbf{p}}_i(t) - \left(\mathbf{p}_i(t) - \mathbf{p}_j(t)\right)^T \dot{\mathbf{p}}_j(t) = \\
&= \mathbf{w}_{ij} \underbrace{\begin{pmatrix} -\mathbf{0}^T - & \underbrace{\left(\mathbf{p}_i(t) - \mathbf{p}_j(t)\right)^T}_{\text{vertex } i} & -\mathbf{0}^T - & \underbrace{\left(\mathbf{p}_j(t) - \mathbf{p}_i(t)\right)^T}_{\text{vertex } j} & -\mathbf{0}^T - \end{pmatrix}}_{K_{ij} \in \mathbb{R}^{1 \times 3N}} \begin{pmatrix} \dot{\mathbf{p}}_1 \\ \vdots \\ \dot{\mathbf{p}}_N \end{pmatrix}
\end{aligned}
$$

where $\mathbf{0} = \begin{pmatrix} 0 & 0 & \dots & 0 \end{pmatrix}^T$

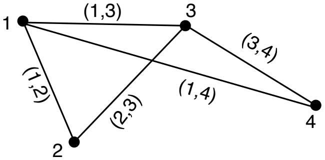
stacking the previous constraints for every $(i, j) \in \{e_1,\ e_2 \ldots e_{N-1} \ldots \ldots, e_{N(N-1)/2}\}$:

$$\underbrace{\begin{pmatrix} \mathbf{w}_{12} & & \\ & \ddots & \\ & & \mathbf{w}_{N(N-1)} \end{pmatrix}}_{W(\mathbf{w}) \in \mathbb{R}^{\frac{N(N-1)}{2} \times \frac{N(N-1)}{2}}} \underbrace{\begin{pmatrix} K_{12} \\ \vdots \\ K_{N(N-1)} \end{pmatrix}}_{K(\mathbf{p})} \underbrace{\begin{pmatrix} \dot{\mathbf{p}}_1 \\ \vdots \\ \dot{\mathbf{p}}_N \end{pmatrix}}_{\dot{\mathbf{p}} \in \mathbb{R}^{3N}} = \underbrace{W(\mathbf{w}) K(\mathbf{p})}_{R(\mathbf{w},\mathbf{p}) \in \mathbb{R}^{\frac{N(N-1)}{2} \times 3N}} \quad \dot{\mathbf{p}} = R(\mathbf{w}, \mathbf{p}) \dot{\mathbf{p}} = \mathbf{0}$$

Rigidity Matrix

$R(\mathbf{w}, \mathbf{p})$ is the (weighted) **rigidity matrix**

Example of Rigidity Matrix



$d = 2$ $(\mathbb{R}^2)$
$N = 4$
$N(N-1)/2 = 6$

$$R(\mathbf{w},\mathbf{p}) =$$

$$\begin{pmatrix}
\mathbf{w}_{12}(p_1^x - p_2^x) & \mathbf{w}_{12}(p_1^y - p_2^y) & \mathbf{w}_{12}(p_2^x - p_1^x) & \mathbf{w}_{12}(p_2^y - p_1^y) & 0 & 0 & 0 & 0 \\
\mathbf{w}_{13}(p_1^x - p_3^x) & \mathbf{w}_{13}(p_1^y - p_3^y) & 0 & 0 & \mathbf{w}_{13}(p_3^x - p_1^x) & \mathbf{w}_{13}(p_3^y - p_1^y) & 0 & 0 \\
\mathbf{w}_{14}(p_1^x - p_4^x) & \mathbf{w}_{14}(p_1^y - p_4^y) & 0 & 0 & 0 & 0 & \mathbf{w}_{14}(p_4^x - p_1^x) & \mathbf{w}_{14}(p_4^y - p_1^y) \\
0 & 0 & \mathbf{w}_{23}(p_2^x - p_3^x) & \mathbf{w}_{23}(p_2^y - p_3^y) & \mathbf{w}_{23}(p_3^x - p_2^x) & \mathbf{w}_{23}(p_3^y - p_2^y) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbf{w}_{34}(p_3^x - p_4^x) & \mathbf{w}_{34}(p_3^y - p_4^y) & \mathbf{w}_{34}(p_4^x - p_3^x) & \mathbf{w}_{34}(p_4^y - p_3^y)
\end{pmatrix}$$

Rigidity vs Infinitesimal Rigidity

- rigidity is defined **combinatorially** ("...s.t. every other framework...")
- infinitesimal rigidity implies rigidity
- converse not true (degenerate cases) but...
- infinitesimal rigidity can be defined **algebraically**, in fact...

- **collective roto-translations** in $\mathbb{R}^3$ keep constant all the distances, by definition, i.e., if $\dot{\mathbf{p}}$ is trivial then $R(\mathbf{w}, \mathbf{p})\dot{\mathbf{p}} = 0$
- $\Rightarrow \text{Dim}\,(\ker[R(\mathbf{w}, \mathbf{p})]) \geq 6$ always

- for infinitesimally rigid frameworks the motion that keep constant all the distances are only **collective roto-translations** in $\mathbb{R}^3$
  i.e., if $R(\mathbf{w}, \mathbf{p})\dot{\mathbf{p}} = 0$ then $\dot{\mathbf{p}}$ is trivial
- infinitesimally rigidity $\Rightarrow \text{Dim}\,(\ker[R(\mathbf{w}, \mathbf{p})]) = 6$

(Tay and Whiteley 1985) and (Zelazo et al. 2014)

A framework is infinitesimally rigid if and only if $\text{rank}[R(\mathbf{w}, \mathbf{p})] = 3N - 6$

- despite its name, the rigidity matrix is actually characterizing **infinitesimal rigidity** (rather than **rigidity**)

$S(\mathbf{w},\mathbf{p}) = R(\mathbf{w},\mathbf{p})^T R(\mathbf{w},\mathbf{p}) \in \mathbb{R}^{3N\times 3N}$ is the **symmetric rigidity matrix**

(Zelazo et al. 2014)

Properties:

P.1 $S = S(\mathbf{w},\mathbf{p}) = S(\mathbf{x}_1,\ldots,\mathbf{x}_N,\mathbf{z})$

P.2 $S \in \mathbb{R}^{3N\times 3N}$ (square)

P.3 $S_{ij} = S_{ji}$ (symmetric)

P.4 $\mathrm{Dim}\left(\ker[S(\mathbf{w},\mathbf{p})]\right) \geq 6$

(Zelazo et al. 2014)

A framework is infinitesimally rigid if and only if $\mathrm{rank}[S(\mathbf{w},\mathbf{p})] = 3N - 6$

Additional properties of $S = R^T R$

- S is **positive semi-definite**, i.e., all the eigenvalues are real and non-negative

$$0 \leq \varsigma_1 \leq \varsigma_2 \leq \ldots \leq \varsigma_6 \leq \varsigma_7 \leq \ldots \leq \varsigma_{3N}$$

- $\mathrm{Dim}\left(\ker[S(\mathbf{w}, \mathbf{p})]\right) \geq 6$, therefore

$$\varsigma_1 = \varsigma_2 = \varsigma_3 = \varsigma_4 = \varsigma_5 = \varsigma_6 = 0$$

(Zelazo et al. 2014)

$\varsigma_7 > 0$ if the framework is **infinitesimally rigid** and $\varsigma_7 = 0$ otherwise

ς_7 provides an **algebraic definition** of infinitesimal rigidity

$\Rightarrow$ ς_7 is called the **rigidity eigenvalue** (Zelazo et al. 2014)

$\varsigma_7 = \varsigma_7(\mathbf{x}_1, \ldots, \mathbf{x}_N, \mathbf{z})$ is a **global** quantity

Connectivity vs Infinitesimal Rigidity

Connectivity

$\exists$ a path between any pair of vertexes

- depends on $\mathbf{x}_1, \ldots, \mathbf{x}_N, \mathbf{z}$ (global property)
- Laplacian matrix $L \in \mathbb{R}^{N \times N}$
- $\Leftrightarrow$ Fidler eigenvalue $\lambda_2 > 0$

Infinitesimal rigidity

distance-preservation on the edges forces a trivial (roto-translational) movement

- depends on $\mathbf{x}_1, \ldots, \mathbf{x}_N, \mathbf{z}$ (global property)
- symmetric rigidity matrix $S \in \mathbb{R}^{3N \times 3N}$
- $\Leftrightarrow$ rigidity eigenvalue $\varsigma_7 > 0$

Rigidity implies Connectivity

(Infinitesimal) Rigidity $\Rightarrow$ Connectivity, i.e., $\varsigma_7 > 0 \Rightarrow \lambda_2 > 0$

In fact, e.g., by contradiction:

- not connected implies at least **two connected components**
- **distance** between the two connected components **can change** still preserving equivalence

$\Rightarrow$ by enforcing infinitesimal rigidity one enforces connectivity as well

Connectivity

- applicable to any graph
- depends only on **w**
- $\not\Rightarrow$ infinitesimal rigidity

Infinitesimal rigidity

- applicable only to frameworks (graphs + positions)
- depends both on **w** and $\mathbf{v} = \mathbf{p}$
- $\Rightarrow$ connectivity

Infinitesimal rigidity is a **stronger property**
and
applies to a **more particular** structure (framework)

Maintenance Problems and Methods

Maintenance Problems

Assume each robot $i = 1, \ldots, N$

- can **control** $\mathbf{x}_i(t)$, $\forall t \geq t_0$ (with $\mathbf{x}_i(t)$ smooth enough)
- has some objectives (**mission**)

Maintenance problem(s)

- assume $\mathcal{G}$ is connected (or $(\mathcal{G}, \mathbf{p})$ is infinitesimally rigid) for $t = t_0$
- control $\mathbf{x}_1(t), \ldots, \mathbf{x}_N(t)$ such that
  1. $\mathcal{G}$ **stays connected** (or $(\mathcal{G}, \mathbf{p})$ **stays infinitesimally rigid**) $\forall t > t_0$
  2. the mission of each robot is accomplished

Maintenance $\neq$

- eventual achievement
- periodical achievement

Using the **algebraic formulation** of connectivity and infinitesimal rigidity

Connectivity maintenance

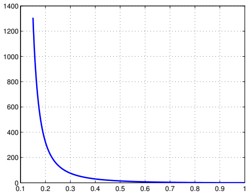
- assume $\lambda_2(t_0) > 0$
- for $t > t_0$
  - **maintain** $\lambda_2(\mathbf{x}_1(t), \dots, \mathbf{x}_N(t), \mathbf{z}) > 0$
  - and accomplish the mission

Infinitesimal rigidity maintenance

- assume $\varsigma_7(t_0) > 0$
- for $t > t_0$
  - **maintain** $\varsigma_7(\mathbf{x}_1(t), \dots, \mathbf{x}_N(t), \mathbf{z}) > 0$
  - and accomplish the mission

Assume robot i can control $\mathbf{x}_i^{(h)} = \frac{\mathrm{d}^h}{\mathrm{d}t^h}\mathbf{x}_i$ for a certain $h \geq 1$

1. define **potential function** $V : (\mu_{\min}, +\infty) \rightarrow \mathbb{R}^+$, that
   - grows unbounded as $\mu \rightarrow^+ \mu^{\min} > 0$
   - vanishes (with vanishing derivatives) as $\mu > \mu^0 > \mu^{\min}$
   - is, at least, C^1, i.e., it exists $\frac{\mathrm{d}V}{\mathrm{d}\mu}$, $\forall \mu > \mu^{\min}$



2. let each robot **command**

$$\mathbf{x}_i^{(h)} = \left.\frac{\mathrm{d}V}{\mathrm{d}\mu}\right|_{\lambda_2(t)} \left.\frac{\partial \lambda_2}{\partial \mathbf{x}_i}\right|_{(\mathbf{x}_1,\ldots,\mathbf{x}_N,\mathbf{z})} + u_i$$

(for **connectivity maintenance**)

$$\mathbf{x}_i^{(h)} = \left.\frac{\mathrm{d}V}{\mathrm{d}\mu}\right|_{\varsigma_7(t)} \left.\frac{\partial \varsigma_7}{\partial \mathbf{x}_i}\right|_{(\mathbf{x}_1,\ldots,\mathbf{x}_N,\mathbf{z})} + u_i$$

(for **infinitesimal rigidity maintenance**)

where u_i is a properly designed additional control input accounting for

- **accomplishment of mission**
- **stability**

connectivity maintenance

$$\frac{\mathrm{d}V}{\mathrm{d}\mu}\bigg|_{\lambda_2(t)} \frac{\partial \lambda_2}{\partial \mathbf{x}_i}\bigg|_{(\mathbf{x}_1,\dots,\mathbf{x}_N,\mathbf{z})}$$

infinitesimal rigidity maintenance

$$\frac{\mathrm{d}V}{\mathrm{d}\mu}\bigg|_{\varsigma_7(t)} \frac{\partial \varsigma_7}{\partial \mathbf{x}_i}\bigg|_{(\mathbf{x}_1,\dots,\mathbf{x}_N,\mathbf{z})}$$

Gradient computation is composed by **two parts**

First part: computation of $\left.\frac{\mathrm{d}V}{\mathrm{d}\mu}\right|_{\lambda_2(t)}$ (or $\left.\frac{\mathrm{d}V}{\mathrm{d}\mu}\right|_{\varsigma_7(t)}$)

requires that **each robot knows**:

- the function V
- $\lambda_2(t)$ (or $\varsigma_7(t)$)

Second part: Computation of $\left.\frac{\partial \lambda_2}{\partial \mathbf{x}_i}\right|_{(\mathbf{x}_1,\ldots,\mathbf{x}_N,\mathbf{z})}$ (or $\left.\frac{\partial \varsigma_7}{\partial \mathbf{x}_i}\right|_{(\mathbf{x}_1,\ldots,\mathbf{x}_N,\mathbf{z})}$)

requires in general

- the **analytic expression** of the **gradient of** λ_2 (or ς_7) with respect to $\mathbf{x}_i$

Given a matrix M, any eigenvalue can be written as $\mu = \mathbf{u}^T M \mathbf{u}$, where

- $\mathbf{u}$ is a normalized eigenvector associated to μ (i.e., $M\mathbf{u} = \mu\mathbf{u}$ and $\mathbf{u}^T\mathbf{u} = 1$)

Connectivity

$$\lambda_2 = \mathbf{u}^T L \mathbf{u}$$

differentiating, we obtain (Yang et al. 2010)

$$\frac{\partial \lambda_2}{\partial \mathbf{x}_i} = \sum_{(j,h)\in\mathcal{E}} \frac{\partial \mathbf{w}_{jh}}{\partial \mathbf{x}_i}(\mathbf{u}_j - \mathbf{u}_h)^2$$

Infinitesimal rigidity

$$\varsigma_7 = \mathbf{u}^T S \mathbf{u}$$

differentiating, we obtain (Zelazo et al. 2014)

$$\frac{\partial \varsigma_7}{\partial \mathbf{x}_i} = \sum_{(j,h)\in\mathcal{E}} \frac{\partial \mathbf{w}_{jh}}{\partial \mathbf{x}_i}\mathbf{s}_{jh} + \frac{\partial \mathbf{s}_{jh}}{\partial \mathbf{x}_i}\mathbf{w}_{jh}$$

$$\begin{aligned}
\mathbf{s}_{jh} = \Big(&(p_j^x - p_h^x)^2(\mathbf{u}_j^x - \mathbf{u}_h^x)^2 + \\
&(p_j^y - p_h^y)^2(\mathbf{u}_j^y - \mathbf{u}_h^y)^2 + \\
&(p_j^z - p_h^z)^2(\mathbf{u}_j^z - \mathbf{u}_h^z)^2 + \\
&2(p_j^x - p_h^x)(p_j^y - p_h^y)(\mathbf{u}_j^x - \mathbf{u}_h^x)(\mathbf{u}_j^y - \mathbf{u}_h^y) + \\
&2(p_j^x - p_h^x)(p_j^z - p_h^z)(\mathbf{u}_j^x - \mathbf{u}_h^x)(\mathbf{u}_j^z - \mathbf{u}_h^z) + \\
&2\,(p_j^y - p_h^y)(p_j^z - p_h^z)(\mathbf{u}_j^y - \mathbf{u}_h^y)(\mathbf{u}_j^z - \mathbf{u}_h^z)\Big)
\end{aligned}$$

Decentralized Control Law

Decentralized control law

Consider a network of robots performing a **control law**

The control law is **decentralized** if, for each robot i, the **size** of the

- **communication** bandwidth
- **computation** time (per step)
- **memory** used (inputs, outputs, local variables)

depends only on $|\mathcal{N}_i|$ and not on N

- a control law that is not decentralized is not **scalable**

Example of decentralized control law: **consensus**

$$\dot{\mathbf{x}}_i = \sum_{j \in \mathcal{N}_i} (\mathbf{x}_j - \mathbf{x}_i) \quad \forall i$$

The two control laws shown so far, i.e.,

connectivity maintenance

$$\left.\frac{\mathrm{d}V}{\mathrm{d}\mu}\right|_{\mu=\lambda_2} \sum_{(j,h)\in\mathcal{E}} \frac{\partial \mathbf{w}_{jh}}{\partial \mathbf{x}_i}(\mathbf{u}_j - \mathbf{u}_h)^2$$

infinitesimal rigidity maintenance

$$\left.\frac{\mathrm{d}V}{\mathrm{d}\mu}\right|_{\mu=\varsigma_7} \sum_{(j,h)\in\mathcal{E}} \frac{\partial \mathbf{w}_{jh}}{\partial \mathbf{x}_i}\mathbf{s}_{jh} + \frac{\partial \mathbf{s}_{jh}}{\partial \mathbf{x}_i}\mathbf{w}_{jh}$$

are **not decentralized** control law because

- each robot must know λ_2 (or ς_7) that depends on $\mathbf{x}_1(t), \ldots, \mathbf{x}_N(t), \mathbf{z}$
- each robot must know $\mathbf{w}_{jh}$ and $\mathbf{s}_{jh}$, $\forall (j,h) \in \mathcal{E}$, and $\mathbf{u}_1, \ldots, \mathbf{u}_N$ that also depend on $\mathbf{x}_1(t), \ldots, \mathbf{x}_N(t), \mathbf{z}$

Goal: make the control law **decentralized**

Locality assumption for the connection map w

$\forall i \in \mathcal{V}, \forall (j,h) \in \mathcal{E} \quad \dfrac{\partial \mathbf{w}_{jh}}{\partial \mathbf{x}_i} = 0$ if neither $j = i$ nor $h = i$

Consequence for connectivity gradient

$$\frac{\partial \lambda_2}{\partial \mathbf{x}_i} = \sum_{(j,h)\in\mathcal{E}} \frac{\partial \mathbf{w}_{jh}}{\partial \mathbf{x}_i}(\mathbf{u}_j - \mathbf{u}_h)^2 = \sum_{j\in\mathcal{N}_i} \frac{\partial \mathbf{w}_{ij}}{\partial \mathbf{x}_i}(\mathbf{u}_i - \mathbf{u}_j)^2$$

$$\frac{\partial \lambda_2}{\partial \mathbf{x}_i} = \sum_{j\in\mathcal{N}_i} \mathbf{f}_\lambda \left(\frac{\partial \mathbf{w}_{ij}}{\partial \mathbf{x}_i}, \mathbf{w}_{ij}, \mathbf{x}_i, \mathbf{x}_j, \mathbf{u}_i, \mathbf{u}_j \right)$$

Locality assumption for the connection map w

$$\forall i \in \mathcal{V},\ \forall (j,h) \in \mathcal{E} \quad \frac{\partial \mathbf{w}_{jh}}{\partial \mathbf{x}_i} = 0 \text{ if neither } j = i \text{ nor } h = i$$

Consequence for infinitesimal rigidity gradient

$$\frac{\partial \varsigma_7}{\partial \mathbf{x}_i} = \sum_{(j,h)\in\mathcal{E}} \frac{\partial \mathbf{w}_{jh}}{\partial \mathbf{x}_i}\mathbf{s}_{jh} + \frac{\partial \mathbf{s}_{jh}}{\partial \mathbf{x}_i}\mathbf{w}_{jh} = \sum_{j\in\mathcal{N}_i} \frac{\partial \mathbf{w}_{ij}}{\partial \mathbf{x}_i}\mathbf{s}_{ij} + \frac{\partial \mathbf{s}_{ij}}{\partial \mathbf{x}_i}\mathbf{w}_{ij} =$$

$$\sum_{j\in\mathcal{N}_i} \frac{\partial \mathbf{w}_{ij}}{\partial \mathbf{x}_i}\Big((p_{ij}^x)^2(\mathbf{u}_i^x - \mathbf{u}_j^x)^2 + (p_{ij}^y)^2(\mathbf{u}_i^y - \mathbf{u}_j^y)^2 + (p_{ij}^z)^2(\mathbf{u}_i^z - \mathbf{u}_j^z)^2 +$$

$$2p_{ij}^x p_{ij}^y(\mathbf{u}_i^x - \mathbf{u}_j^x)(\mathbf{u}_i^y - \mathbf{u}_j^y) + 2p_{ij}^x p_{ij}^z(\mathbf{u}_i^x - \mathbf{u}_j^x)(\mathbf{u}_i^z - \mathbf{u}_j^z) + 2p_{ij}^y p_{ij}^z(\mathbf{u}_i^y - \mathbf{u}_j^y)(\mathbf{u}_i^z - \mathbf{u}_j^z)\Big)$$

$$+ \sum_{j\in\mathcal{N}_i} \begin{pmatrix} \mathbf{u}_i^x - \mathbf{u}_j^x \\ \mathbf{u}_i^y - \mathbf{u}_j^y \\ \mathbf{u}_i^z - \mathbf{u}_j^z \end{pmatrix} 2\mathbf{w}_{ij} \left(p_{ij}^x(\mathbf{u}_i^x - \mathbf{u}_j^x) + p_{ij}^y(\mathbf{u}_i^y - \mathbf{u}_j^y) + p_{ij}^z(\mathbf{u}_i^z - \mathbf{u}_j^z)\right)$$

$$\frac{\partial \varsigma_7}{\partial \mathbf{x}_i} = \sum_{j\in\mathcal{N}_i} \mathbf{f}_\varsigma\left(\frac{\partial \mathbf{w}_{ij}}{\partial \mathbf{x}_i}, \mathbf{w}_{ij}, \mathbf{x}_i, \mathbf{x}_j, \mathbf{u}_i, \mathbf{u}_j\right)$$

where $p_{ij} = p_i - p_j$

Locality assumption for the connection map w

$\forall i \in \mathcal{V}, \forall (j,h) \in \mathcal{E} \quad \dfrac{\partial \mathbf{w}_{jh}}{\partial \mathbf{x}_i} = 0$ if neither $j = i$ nor $h = i$

The two gradient-based control laws with **locality assumption**

connectivity maintenance

$$V'(\lambda_2) \sum_{j \in \mathcal{N}_i} \mathbf{f}_\lambda \left(\frac{\partial \mathbf{w}_{ij}}{\partial \mathbf{x}_i}, \mathbf{w}_{ij}, \mathbf{x}_i, \mathbf{x}_j, \mathbf{u}_i, \mathbf{u}_j \right)$$

infinitesimal rigidity maintenance

$$V'(\varsigma_7) \sum_{j \in \mathcal{N}_i} \mathbf{f}_\varsigma \left(\frac{\partial \mathbf{w}_{ij}}{\partial \mathbf{x}_i}, \mathbf{w}_{ij}, \mathbf{x}_i, \mathbf{x}_j, \mathbf{u}_i, \mathbf{u}_j \right)$$

become **partially decentralized** control law, each robot must know:

- λ_2 (or ς_7) that depends on $\mathbf{x}_1(t), \ldots, \mathbf{x}_N(t), \mathbf{z}$ (**not decentralized**)
- $\mathbf{x}_i$, $\mathbf{w}_{ij}$, $\dfrac{\partial \mathbf{w}_{ij}}{\partial \mathbf{x}_i}$, and $\mathbf{x}_j$, $\forall j \in \mathcal{N}_i$, and $\mathbf{z}$, (**decentralized**)
- $\mathbf{u}_i$ and $\mathbf{u}_j$, $\forall j \in \mathcal{N}_i$ that depend on $\mathbf{x}_1(t), \ldots, \mathbf{x}_N(t), \mathbf{z}$ (**not decentralized**)

Goal: compute λ_2 (or ς_7), $\mathbf{u}_i$ and $\mathbf{u}_j$ $\forall j \in \mathcal{N}_i$ in a decentralized way

Computation of λ_2 and ς_7

Continuous power iteration method (Yang et al. 2010; Zelazo et al. 2014)

An **iterative algorithm** to get an estimate $\hat{\mu}$ and $\hat{\mathbf{u}}$ of the the l-th eigenvalue μ and the associated eigenvector $\mathbf{u}$ of a positive semidefinite matrix $M \in \mathbb{R}^n$

Denote with $T \in \mathbb{R}^{n \times l-1}$ the image matrix of the first $l-1$ eigenvectors

$$\dot{\hat{\mathbf{u}}} = -k_1 TT^T\hat{\mathbf{u}} - k_2 M\hat{\mathbf{u}} - k_3\left(\frac{\hat{\mathbf{u}}^T\hat{\mathbf{u}}}{n} - 1\right)$$

- $-k_1 TT^T\hat{\mathbf{u}}$: **deflation**: to remove the components spanned by the first $l-1$ eigenvectors
- $-k_2 M\hat{\mathbf{u}}$: **direction update**, to move towards $\mathbf{u}$
- $-k_3\left(\frac{\hat{\mathbf{u}}^T\hat{\mathbf{u}}}{n} - 1\right)$: **renormalization** to stay away from the null vector

The **eigenvalue** is estimated as

$$\hat{\mu} = \frac{k_3}{k_2}\left(1 - \|\hat{\mathbf{u}}\|^2\right)$$

Decentralized power iteration method (Yang et al. 2010; Zelazo et al. 2014)

$$\dot{\hat{\mathbf{u}}} = -k_1 T T^T \hat{\mathbf{u}} - k_2 M \hat{\mathbf{u}} - k_3 \left(\frac{\hat{\mathbf{u}}^T \hat{\mathbf{u}}}{n} - 1 \right)$$

connectivity maintenance

$$M = L$$
$$T = \mathbf{1}$$

infinitesimal rigidity maintenance

$$M = S$$

$T \in \mathbb{R}^{3N \times 6}$ def. in (Zelazo et al. 2014)

The only remaining global quantities

- $T^T \hat{\mathbf{u}}$
- $\hat{\mathbf{u}}^T \hat{\mathbf{u}}$

can be estimated using the
proportional/integral-average consensus estimator (PI-ACE) (Yang et al. 2010)

Possible **limits** of the gradient-based methods

- the robot could be **unable to follow** the gradient because of, e.g, input saturation
- possibility of **local minima** (depending on the environment complexity)

Possible **limits** of the decentralized methods:

- need for **time-scale separation**:
  decentralized estimator dynamics must be faster than motion control dynamics
- the **gains** of the decentralized estimator must be carefully tuned depending on N
- decentralized power iteration does not work for eigenvalues with **multiplicity** > 1
- (decentralized) power iteration has a relatively **slow convergence**

Possible destabilization due to non-perfect estimation can be mitigated using **passivity theory** (Robuffo Giordano et al. 2013)

Handling Multiple Objectives in Maintenance Problems

Communication and Sensing Objectives

Connectivity in a network of robots is typically associated to

Inter-robot

- **communication**
- **relative sensing**

Quality of inter-robot sensing/communication modeled by a sufficiently **smooth non-negative scalar** function

$$\gamma_{ij} = \gamma(\mathbf{x}_i, \mathbf{x}_j, \mathbf{z}) \geq 0$$

Measures the **quality** of the **mutual information exchange**

- $\gamma_{ij} = 0$ if no exchange is possible and
- $\gamma_{ij} > 0$ otherwise
- the larger γ_{ij} the better the quality

Straightforward use:

$$\mathbf{w}_{ij} = \gamma_{ij}$$

In order to handle multiple objectives define

$$\mathbf{w}_{ij} = \alpha_{ij}\beta_{ij}\gamma_{ij}$$

where

- $\alpha_{ij} \geq 0$ encodes **hard constraints**
- $\beta_{ij} \geq 0$ encodes **soft requirements**
- $\gamma_{ij} \geq 0$ encodes the **communication/sensing objectives** (defined before)

this defines the

- **generalized connectivity**, and a
- **generalized infinitesimal rigidity**

Hard constraints: conditions $\text{HD}_1, \text{HD}_2, \ldots$ that must be **true** $\forall t \geq 0$

Maintenance methods **automatically** keep true a hard constraint: **$\text{HD}_0 \equiv$ connectivity**

Idea: define α_{ij} such that

- not HD_h for some $h \Rightarrow \text{not HD}_0$

How? Just define α_{ij} s.t.

- not HD_h for some $h \Rightarrow \alpha_{ij} = 0, \forall j = 1, \ldots, N$

Why only $\alpha_{ij} = 0, \forall j = 1, \ldots, N$?

- it is enough for non-connectivity
  ($\alpha_{ij} = 0, \forall j = 1, \ldots, N$ implies robot i becomes disconnected from the rest)
- is intrinsically decentralized

α_{ij} must be smooth enough to allow for gradient computation

- the more $\alpha_{ij} \to 0$ the closer to not HD_h

Soft requirements: should be **preferably** realized by the individual pair (i, j)

Notes:

- gradient-based maintenance methods tend to **maximize** the **maintenance eigenvalues** (e.g., λ_2 or ς_7)
- maintenance eigenvalues monotonically increase w.r.t. $\mathbf{w}_{ij}$ $\forall (i, j) \in \mathcal{E}$

Idea: define β_{ij} such that

- has a unique maximum when the soft constraints are realized
- monotonically decreases down to $\beta_{ij} = 0$ otherwise

Non-perfect compliance with a soft requirement leads to

- corresponding decrease of maintenance eigenvalue
  $\downarrow \beta_{ij} \Rightarrow \downarrow \mathbf{w}_{ij} \Rightarrow \downarrow \lambda_2$ (or $\downarrow \varsigma_7$)

Complete violation of soft requirement

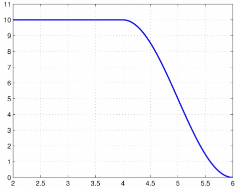
- leads to disconnected edge (i, j), but
- does not (in general) result in a global loss of connectivity for the graph

Applications

Communication/sensing objectives $\rightarrow \gamma_{ij}(\mathbf{x}_i, \mathbf{x}_j, \mathbf{z})$

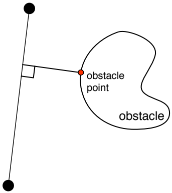
Proximity sensing model:

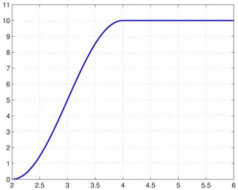
- $D > 0$ is a suitable sensing/communication **maximum range** (e.g, radio signal)
- robot i and j able to interact iff $\|\mathbf{x}_i - \mathbf{x}_j\| < D$,



Proximity-visibility sensing model (e.g., onboard cameras):

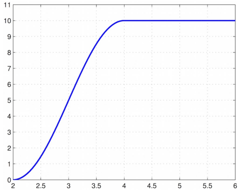
- S_{ij} **line-of-sight** segment joining $\mathbf{x}_i$ and $\mathbf{x}_j$
- robot i and j able to interact iff $\|\mathbf{x}_i - \mathbf{x}_j\| < D$, and $\mathrm{dist}\left(S_{ij(\mathbf{x}_i,\mathbf{x}_j)}, \mathrm{obst}(\mathbf{z})\right) > D_{\mathrm{vis}}$





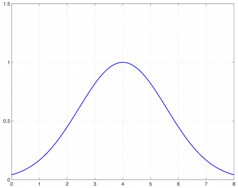
Hard constraints $\rightarrow \alpha_{ij}$

e.g., **inter-robot collision avoidance**:
$\|\mathbf{x}_i - \mathbf{x}_j\| > d_0$



Soft requirements $\rightarrow \beta_{ij}$

e.g., **formation control**, e.g.,
$\|\mathbf{x}_i - \mathbf{x}_j\| \simeq d_{\mathrm{des}}$



Mission: **concurrent exploration** of a **sequence** of **targets**
While maintaining "generalized" **connectivity**, i.e., including

- proximity/visibility sensing model
- collision avoidance
- preferred inter-distance

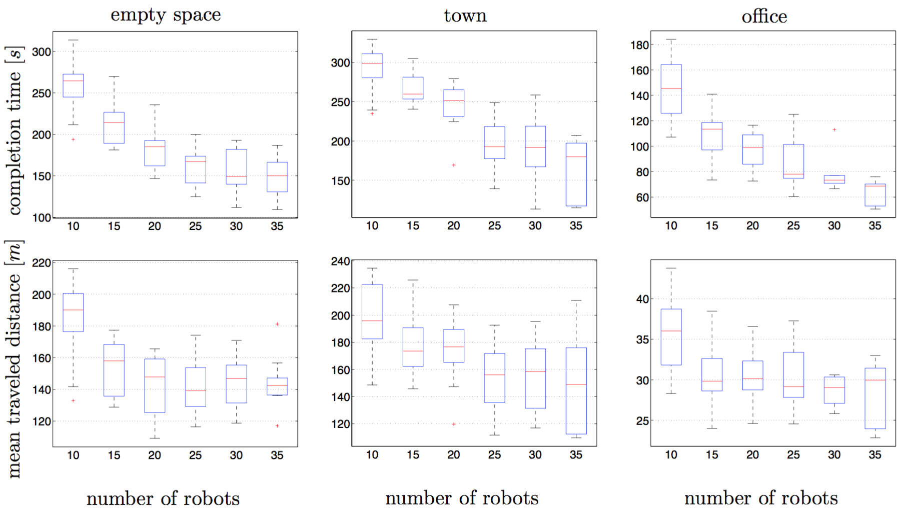
Connectivity maintenance in case of, e.g., **second order** systems:

$$\ddot{\mathbf{x}}_i = \left.\frac{\mathrm{d}V}{\mathrm{d}\mu}\right|_{\lambda_2(t)} \left.\frac{\partial \lambda_2}{\partial \mathbf{x}_i}\right|_{(\mathbf{x}_1,\ldots,\mathbf{x}_N,\mathbf{z})} + u_i$$

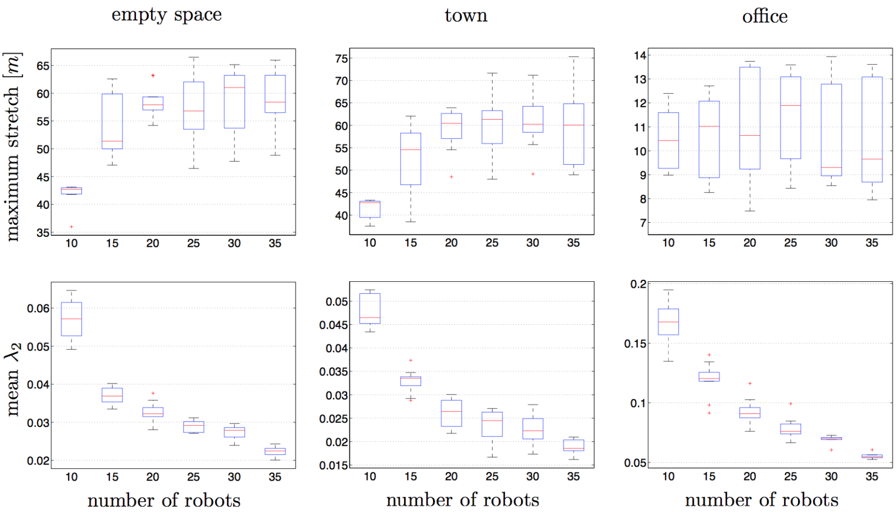
$$u_i = -B\dot{\mathbf{x}}_i + f_i^{\mathrm{expl}}$$

- $-B\dot{\mathbf{x}}_i$ stabilizing damping
- f_i^{expl} multi-target exploration force (Nestmeyer et al. 2015, Under Review)

videos: `http://homepages.laas.fr/afranchi/videos/multi_exp_conn.html`


empty space
town
office
completion time [s]
mean traveled distance [m]
number of robots
number of robots
number of robots

Multi-Target Exploration with Connectivity



Mission: **unilateral multi-user teleoperation** of some robots in the team
While maintaining "generalized" **infinitesimal rigidity**, i.e., including

- proximity/visibility sensing model
- collision avoidance
- preferred inter-distance

Infinitesimal rigidity maintenance in case of, e.g., **first order** systems:

$$\dot{\mathbf{x}}_i = \left.\frac{\mathrm{d}V}{\mathrm{d}\mu}\right|_{\varsigma_7(t)} \left.\frac{\partial \varsigma_7}{\partial \mathbf{x}_i}\right|_{(\mathbf{x}_1,\ldots,\mathbf{x}_N,\mathbf{z})} + u_i$$

$$u_i = \begin{cases} v_i^h & \text{if connected to a human} \\ 0 & \text{otherwise} \end{cases}$$

- v_i^h **desired velocity** commanded by a **human**

videos: `http://homepages.laas.fr/afranchi/robotics/?q=node/134`

Short summary

- Single **scalars** can define **fundamental global properties**
  - λ_2 Fiedler eigenvalue (Fiedler 1973)
  - ς_7 rigidity eigenvalue (Zelazo et al. 2014)
- **Distributed** computation of the **gradient** is possible
  - \+ smooth
  - \+ online computation (fast)
  - \- presence of local minima

Some open problems

- coinciding eigenvalues
- local minima (using decentralized global planning?)

Decentralized **multi-target exploration** with connectivity maintenance

- T. Nestmeyer, Robuffo Giordano, P., Bülthoff, H. H., and Franchi, A., "Decentralized Simultaneous Multi-target Exploration using a Connected Network of Multiple Robots", Under Review.

Bearing rigidity (in $SE(3)$)

- D. Zelazo, Franchi, A., and Robuffo Giordano, P., "Rigidity Theory in SE(2) for Unscaled Relative Position Estimation using only Bearing", in 2014 European Control Conference, Strasbourg, France, 2014, pp. 2703-2708.
- D. Zelazo, Robuffo Giordano, P., and Franchi, A., "Bearing-Only Formation Control Using an SE(2) Rigidity Theory", in 54rd IEEE Conference on Decision and Control, Osaka, Japan, 2015

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Questions?

Connectivity, Rigidity and Online Decentralized Maintenance Methods

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